Propagator, Green's function and Correlation

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1 Green's function

Consider a linear differential equation written in the general form

$$L(x)u(x) = f(x) \tag{1}$$

where $L(x)$ is a linear, self-adjoint differential operator, $u(x)$ is the unknown function, and $f(x)$ is a known non-homogeneous term. We can write a solution to Eq. 1 as

$$u(x) = L^{-1}f(x) \equiv \int G(x, x')f(x')dx'. \tag{2}$$

Now

$$L(x)u(x) = \int L(x)G(x, x')f(x')dx' = f(x). \tag{3}$$

Thus

$$L(x)G(x, x') = \delta(x - x'). \tag{4}$$

2 Green's function in electrostatics

The basic equation for electrostatics is the Poisson equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}, \tag{5}$$

where Φ is the potential and ρ is charge density. In free space, we know that

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3x'. \tag{6}$$

So the potential at $\mathbf{x}$ due to a unit point charge at $\mathbf{x}''$, $\rho(\mathbf{x}') = -4\pi\epsilon_0\,\delta(\mathbf{x}'' - \mathbf{x}')$ is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3x' = -\frac{1}{|\mathbf{x} - \mathbf{x}''|}. \tag{7}$$

Or, due the singular nature of the Laplacian of $1/r$, (the integral form of the Guass's theorem), $\nabla^2(1/r) = 0$ for $r \neq 0$ and its volume integral is -4π, we can write the formal equation,

$$\nabla^2(1/r) = -4\pi\delta(\mathbf{x} - \mathbf{0}), \tag{8}$$

or, more generally,

$$\nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x}-\mathbf{x}'), \tag{9}$$

The free-space Green's function for the three-variable Laplace equation is given in terms of the reciprocal distance between two points. That is to say the solution of the equation

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{10}$$

is

$$G(\mathbf{x},\mathbf{x}') = -\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{11}$$

This green's function is the potential due to a charge at $\mathbf{x}'$. So the potential is

$$\Phi(\mathbf{x}) = \int G(\mathbf{x},\mathbf{x}') \cdot \rho(\mathbf{x}') = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'. \tag{12}$$

With boundary, the Green's function becomes

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}') \tag{13}$$

with the function F satisfying the Laplace equation inside the volumn V

3 Eigenfunction expansion for Green's Function

For, the Laplace equation in spherical coordinates (r, θ, ϕ), when the system has azimuthal symmetry, i.e. the solution is independent of ϕ, the general solution is:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta). \tag{14}$$

Then the Green's function can be expanded at $\mathbf{x}'$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) \tag{15}$$

Due to boundary conditions, the system has eigenfunctions! For example, using separation of variables, we can solve the Laplace equation in a rectangular box with boundary conditions. Then the solution is the superposition of the eigenfunctions (normal modes, standing waves in this case).

For example, for an elliptic differential equation of the form

$$\nabla^2 \psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda]\psi(\mathbf{x}) = 0, \tag{16}$$

with homogeneous boundary conditions, the system can only has well-behaved solutions for certain values of λ. These values of λ, denoted by λ_n, are called eigenvalues and the solutions $\psi_n(\mathbf{x})$ are called eigenfuctions. The eigenvalue differential equation in written:

$$\nabla^2 \psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0, \tag{17}$$

The eigenfunction are orthogonal. Suppose now we wish to find the Green's function for the equation:

$$\nabla_x^2 G(\mathbf{x}, \mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \tag{18}$$

with the same boundary conditions, where λ is not equal to one of the eigenvalues λ_n. Then the Green's function can be expanded in a series of the eigenfunctions of the form:

$$G(\mathbf{x}, \mathbf{x}') = \sum_n a_n(\mathbf{x}')\psi_n(\mathbf{x}). \tag{19}$$

Substitution into the differential equation for the Green function leading to:

$$\sum_m a_m(\mathbf{x}')(\lambda - \lambda_m)\psi_m(\mathbf{x}) = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \tag{20}$$

Multiply both sides by $\psi_n^*(\mathbf{x})$ and integrate over the volume using the orthogonality conditions, we have

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n - \lambda} \tag{21}$$

For Poisson equation, $f(\mathbf{x}) = 0$ and $\lambda = 0$, we let the eigen Eq. 17 equation be the wave equation over all spaces:

$$(\nabla^2 + k^2)\psi_{\mathbf{k}}(\mathbf{x}) = 0 \tag{22}$$

with the continuum of eigenvalues, k^2, and the eigenfunctions:

$$\psi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{23}$$

Then, according to Eq. 21, the infinite space Green's function has the expansion:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{2\pi^2} \int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2} \tag{24}$$

4 Green's function for wave equation

From the Maxwell equations, we can derive the wave equations:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{25}$$

and

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} \tag{26}$$

We consider the basic structure

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\mathbf{x}, t) \tag{27}$$

For simple situation of no boundary surfaces, suppose that $\Psi(\mathbf{x},t)$ and $f(\mathbf{x},t)$ have the Fourier integral representations,

$$\Psi(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\mathbf{x},t)e^{-i\omega t}d\omega$$
$$f(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\mathbf{x},t)e^{-i\omega t}d\omega. \tag{28}$$

Inserting the above equation into the wave equation, the Fourier transform $\Psi(\mathbf{x},\omega)$ satisfies the inhomogeneous Helmholtz wave equation

$$(\nabla^2 + k^2)\Psi(\mathbf{x},\omega) = -4\pi f(\mathbf{x},\omega) \tag{29}$$

for each value of ω. Here $k = \omega/c$ is the wave number associated with frequency ω. If there are no boundary surfaces, the Green function can depend only on $R = |\mathbf{R}| = |\mathbf{x}-\mathbf{x}'|$. The Green's function satisfying

$$(\nabla^2 + k^2)G_k(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{30}$$

is

$$G_k(R) = AG_k^{(+)}(R) + BG_k^{(-)}(R) \tag{31}$$

where

$$G_k^{(\pm)}(R) = \frac{e^{\pm ikR}}{R} \tag{32}$$

The first term represents a diverging spherical wave propagating from the origin, while the second represents a converging spherical wave.

To understand the different time behaviors associated with $G_k^{(+)}$ and $G_k^{(-)}$ we need to construct the corresponding time-dependent Green functions that satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G^{(\pm)}(\mathbf{x},t;\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{33}$$

The source term for the Helmholtz equation Eq. 29 is

$$f(\mathbf{x},\omega) = \int_{\infty}^{\infty} f(\mathbf{x},t)e^{i\omega t}dt = \int_{\infty}^{\infty} \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')e^{i\omega t}dt = -4\pi\delta(\mathbf{x}-\mathbf{x}')e^{i\omega t'} \tag{34}$$

So the solution are therefore $G_k^{(\pm)}(R)e^{i\omega t'}$. So, from Eq. 28 the time-dependent Green functions are

$$G^{(\pm)}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ikR}}{R}e^{-i\omega\tau}d\omega \tag{35}$$

where $\tau = t-t'$ is the relative time. For a nondispersive medium where $k = \omega/c$, the integral above is a delta function. So the Green's function are

$$G^{(\pm)}(R,\tau) = \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right) \tag{36}$$

or,

$$G^{(\pm)}(R,\tau) = \frac{\delta\left(t' - \left[t \mp \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right]\right)}{|\mathbf{x}-\mathbf{x}'|} \tag{37}$$

The Green function $G^{(+)}$ is called the retarded Green function because it exhibits the causal behavior associated with a wave disturbance: an effect observed at the point $\mathbf{x}$ at time t is caused by the action of a source a distance R away at an earlier or retarded time, $t' = t - R/c$.

5 propagator

The Schrodinger Equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle . \tag{38}$$

General approach (not in a specific representation).

$$i\hbar |\dot{\psi}(t)\rangle = H |\psi(t)\rangle . \tag{39}$$

Now solve the time-independent Schrodinger equation

$$H |E\rangle = E |E\rangle . \tag{40}$$

Then

$$|\psi(t)\rangle = \sum |E\rangle \langle E|\psi(t)\rangle \equiv \sum a_E(t) |E\rangle . \tag{41}$$

From the Schrodinger Equation, Eq. 39, we have

$$(i\hbar\partial/\partial t - H) |\psi(t)\rangle = \sum (i\hbar \dot{a}_E - E a_E) |E\rangle \tag{42}$$

or

$$i\hbar \dot{a}_E = E a_E \tag{43}$$

Then the solution to a_E is

$$a_E(t) = a_E(0) e^{-iEt/\hbar}. \tag{44}$$

Or,

$$\langle E|\psi(t)\rangle = \langle E|\psi(0)\rangle e^{-iEt/\hbar}. \tag{45}$$

So

$$|\psi(t)\rangle = \sum_E |E\rangle \langle E|\psi(0)\rangle e^{-iEt/\hbar}. \tag{46}$$

Now the propagator $U(t)$ is

$$\boxed{U(t) = \textstyle\sum_E |E\rangle \langle E| e^{-iEt/\hbar}}, \tag{47}$$

and

$$\boxed{|\psi(t)\rangle = U(t) |\psi(0)\rangle} \tag{48}$$

If choose eigenstates in terms of momentum, then

$$U(t) = \int_{-\infty}^{\infty} |p\rangle \langle p| e^{-iE(p)t/\hbar} \, dp \tag{49}$$

Or, actually,

$$U(t, t_0) = e^{-iH(t-t_0)/\hbar} \tag{50}$$

for $t > t_0$!!

Now choose space x representation:

$$\langle x|\psi(t)\rangle = \int_{-\infty}^{\infty} \langle x|U(t)|x'\rangle \langle x'|\psi(0)\rangle \, dx' \tag{51}$$

Or,

$$\psi(x,t) = \int_{-\infty}^{\infty} U(x,t;x',t')\psi(x',t')dx', \tag{52}$$

where $U(x,t;x',t') = \langle x|U(t-t')|x'\rangle$. Suppose we start off with a particle localized at $x' = x'_0$, that is, $\psi(x',t') = \delta(x'-x'_0)$. Then

$$\psi(x,t) = U(x,t;x'_0,t') \tag{53}$$

In other words, the propagator (in the X basis) is the amplitude that a particle starting out at (x'_0,t') ends with at (x,t)

6 Propagator and green's function

Recall Schrodinger Equation in X basis

$$[i\partial_t - H]\,\psi(x,t) = 0. \tag{54}$$

We define the Green's function by

$$[i\partial_t - H]\,G(x,t;x',t) = \delta(x-x')\delta(t-t') \tag{55}$$

with boundary condition $G(x,t;x',t) = 0$, for $t < t_0$. Then the wave function is

$$\psi(x,t) = \int dx' G(x,t;x',t')\psi(x',t'). \tag{56}$$

7 Green's function in scattering

7.1 Definition of many-body Green's function

The scattering problem is to find the full solution of Schrodinger equation

$$(\nabla^2 + k^2)\psi_{\mathbf{k}} = \frac{2\mu}{\hbar^2} V\psi_{\mathbf{k}} \tag{57}$$

for a particle with energy $k^2\hbar^2/2\mu$ One fines the Green's function

$$(\nabla^2 + k^2)G^0(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}') \tag{58}$$

So Green's function is the effect of a unit (δ) potential. The solution will be

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi^0(\mathbf{r}) + \frac{2\mu}{\hbar^2}\int G^0(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')d^3\mathbf{r}' \tag{59}$$

where $\psi^0(\mathbf{r})$ is an arbitary free-particle solution of energy $k^2\hbar^2/2\mu$:

$$(\nabla^2 + k^2)\psi^0 = 0 \tag{60}$$

So the Green's function is the propagation of the perturbation (the potential). Make Fourier transformation:

$$G^0(\mathbf{r}-\mathbf{r}') = \int d^3q\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}G^0(\mathbf{q}) \tag{61}$$

$G^0(\mathbf{q})$ *is the amplitude of the propagation of mode* $e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$ *of* $G^0(\mathbf{r}-\mathbf{r}')$. And

$$\delta^3(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} \tag{62}$$

Using $\nabla^2 e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} = -q^2 e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$, equation 58 becomes

$$(k^2-q^2)G^0(\mathbf{q}) = \frac{1}{(2\pi)^3} \tag{63}$$

Then

$$G^0(\mathbf{q}) = \frac{1}{(2\pi)^3(k^2-q^2)} \tag{64}$$

So

$$\begin{aligned} G^0(\mathbf{r}-\mathbf{r}') &= \frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}G^0(\mathbf{q}) \\ &= \frac{1}{(2\pi)^3}\int d^3q\, \frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(k^2-q^2)} \end{aligned} \tag{65}$$

To make this integral converge, write

$$\begin{aligned} G^0(\mathbf{q}) &= \frac{1}{(2\pi)^3(k^2-q^2+i\epsilon)} \\ & \frac{1}{(2\pi)^3(q-(k+i\delta))(q-(-k-i\delta))} \end{aligned} \tag{66}$$

Make a semi-circuit contour integral in the upper plane, we have

$$\begin{aligned} G^0(\mathbf{r}-\mathbf{r}') &= \lim_{\delta\to 0}\frac{1}{(2\pi)^3}\int d^3q\, \frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(q-(k+i\delta))(q-(-k-i\delta))} \\ &= \lim_{\delta\to 0} 2\pi i Res(k+i\delta) \\ &= \lim_{\delta\to 0} 2\pi i\frac{i}{8\pi^2 r}e^{i(k+i\delta)r} \\ &= -\frac{e^{ikr}}{4\pi r} \end{aligned} \tag{67}$$

8 Many-body Green's function

In the interaction representation, the Hamiltonian is separated into:

$$H = H_0 + V, \tag{68}$$

where the unperturbed part H_0 is exactly solvable. Operators have a time dependence

$$\hat{O}(t) = e^{iH_0t}Oe^{-iH_0t}, \tag{69}$$

and the wave function have a time dependence

$$\hat{\psi}(t) = e^{iH_0t}\psi(t) = e^{iH_0t}e^{-iHt}\psi(0), \tag{70}$$

where $\psi(t)$ is the wave function in the Schrodinger picture. Notice $e^{iH_0 t}e^{-iHt} \neq e^{iH_0 t - iHt}$ unless $[H_0, V] = 0$. In the interaction picture, the wave function can be solved from

$$\frac{\partial}{\partial t}\hat{\psi}(t) = -i\hat{V}(t)\hat{\psi}(t) \tag{71}$$

which leads to the operator $U(t)$:

$$\begin{aligned} U(t) &= e^{iH_0 t}e^{-iHt} \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n T[\hat{V}(t_1)\hat{V}(t_2)\cdots\hat{V}(t_n)] \\ &= T\exp\left[-i\int_0^t dt_1 \hat{V}(t_1)\right] \end{aligned} \tag{72}$$

where T is the time-ordering operator The S matrix is defined by

$$\hat{\psi}(t) = S(t, t')\hat{\psi}(t'). \tag{73}$$

The definition of wave function in interaction picture immediately gives

$$S(t, t') = U(t)U^{\dagger}(t') = T\exp\left[-i\int_{t'}^{t} dt_1 \hat{V}(t_1)\right] \tag{74}$$

Notice $\hat{\psi}(0) = \psi(0)$

At zero temperature the electron Green's function is defined as

$$G(\lambda, t - t') = -i\left\langle |TC_\lambda(t)C_\lambda^{\dagger}(t')| \right\rangle \tag{75}$$

The quantum number λ can be anything depending on the problem of interest. At zero temperature the state $|\rangle$ must be the ground state. If the Hamiltonian of the problem is chosen to be H, then $|\rangle$ is the ground state of H, not H_0, and not known. Write $H = H_0 + V$, choose ground state of H_0. In the definition of the Green's function the C_λ represent states of H_0, while the ground state $|\rangle$ is an eigenstate of H. Furthermore, Eq. 75 is defined in the Heisenberg representation, so that $|\rangle$ is independent of time, while $C_\lambda(t)$ is given by $C_\lambda(t) = e^{iHt}C_\lambda e^{-iHt}$.

To understand the Green's function is to observe that it describes a certain Gedanken experiment. For $t > t'$, one takes the real ground state, and at a time t' one creates an excitation λ. At a later time t one destroys the same excitation. Now if λ were an eigenstate of H, with $HC_\lambda^{\dagger}|\rangle = \varepsilon_\lambda C_\lambda^{\dagger}|\rangle$ and $H|\rangle = \varepsilon_0|\rangle$, then this state would simply propagate with a phase term

$$G(\lambda, t > t') = -i\exp[-i(t - t')(\varepsilon_\lambda - \varepsilon_0)]. \tag{76}$$

Because λ is not usually an eigenstate of H, the particle in the state λ gets scattered, shifted in energy, etc. When one measures at a later time t, to see how much amplitude is left in the state λ, the measurement provides information about the system.

The Green's function can be convert to the interaction representation:

$$G(\lambda, t - t') = -i\frac{{}_0\left\langle |T\hat{C}_\lambda(t)\hat{C}_\lambda^{\dagger}(t')S(\infty, -\infty)| \right\rangle_0}{{}_0\left\langle |TS(\infty, -\infty)| \right\rangle_0} \tag{77}$$

A Green's function can also be defined for the special case where the interactions $V = 0$ and hence the S matrix is unity:

$$G^{(0)}(\lambda, t-t') = -i_0 \langle| T\hat{C}_\lambda(t)\hat{C}^\dagger_\lambda(t')|\rangle_0 \tag{78}$$

There are two quite different types of electronic systems in which we want o employ the Green's function analysis.

1. An empty band. Here the properties are studied of an electron in an energy band in which it is the only electron. An example is an electron in the conduction band of a semiconductor or an insulator. In this case the ground state is the particle vacuum. The unperturbed Green's function is

$$G^{(0)}(\lambda, t-t') = -i\Theta(t-t')e^{-i\varepsilon_\lambda(t-t')} \tag{79}$$

The Fourier transform gives

$$G^{(0)}(\lambda, E) = -i\int_0^\infty dt\, e^{i(E-\varepsilon_\lambda+i\delta)t} = \frac{1}{E-\varepsilon_\lambda+i\delta} \tag{80}$$

2. A degenerate electron gas. The electrons are in a Fermi sea at zero temperature. The standard example is a simple metal. The system has a chemical potential μ, all electron states with $E < \mu$ is occupied. If the unperturbed electrons (eigenstates of H_0) are characterized by an energy $\varepsilon_{\mathbf{k}}$. The ground state $|\rangle_0$ has all states $\varepsilon_{\mathbf{k}} < \mu$ filled and states $\varepsilon_{\mathbf{k}} > \mu$ empty. It is convenient and conventional to measure the electron's energy relative to the chemical potential, to define $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$. For a spherical Fermi surface with Fermi wave vector p_F,

$$_0\langle|C_{\mathbf{k}}C^\dagger_{\mathbf{k}}|\rangle_0 = \Theta(p_F - k) \tag{81}$$

$$_0\langle|C^\dagger_{\mathbf{k}}C_{\mathbf{k}}|\rangle_0 = \Theta(k - p_F) \tag{82}$$

or more generally,

$$_0\langle|C^\dagger_{\mathbf{k}}C_{\mathbf{k}}|\rangle_0 = \Theta(-\xi_{\mathbf{k}}) = \lim_{\beta\to\infty}\frac{1}{e^{\beta\xi_{\mathbf{k}}}+1} \equiv n_F(\xi_{\mathbf{k}}) \tag{83}$$

The unperturbed Green's function is now

$$G^{(0)}(\lambda, t-t') = -i[\Theta(t-t')\Theta(\xi_{\mathbf{k}}) - \Theta(t'-t)\Theta(-\xi_{\mathbf{k}})]e^{-i\xi_{\mathbf{k}}(t-t')} \tag{84}$$

The Fourier transform is

$$G^{(0)}(\mathbf{k}, E) = \frac{\Theta(\xi_{\mathbf{k}})}{E-\xi_{\mathbf{k}}+i\delta} + \frac{\Theta(-\xi_{\mathbf{k}})}{E-\xi_{\mathbf{k}}-i\delta} \tag{85}$$

3. Phonons. The Green's function for phonons is defined as

$$D(\mathbf{q}, \lambda, t-t') = -i\langle|TA_{\mathbf{q}\lambda}(t)A_{-\mathbf{q}\lambda}(t')|\rangle \tag{86}$$

$$A_{\mathbf{q}\lambda} = a_{\mathbf{q}\lambda}(t) + a^\dagger_{-\mathbf{q}\lambda}(t) \tag{87}$$

The subscripts λ refer to the polarization of the phonons. Most applications have one kind of phonon in Hamiltonians which do not mix polarizations. We omit these subscripts for now. At zero temperature there are no phonons. The ground state is again vacuum.

At zero temperature, the unperturbed Green's function for phonon is

$$D^{(0)} = (\mathbf{q}, t - t') = -i[\Theta(t - t')e^{-i\omega_q(t-t')} + \Theta(t' - t)e^{-i\omega_q(t'-t)}]. \tag{88}$$

The Fourier transform is

$$D^{(0)}(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} D^{(0)}(\mathbf{q}, t) = \frac{2\omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2 + i\delta} \tag{89}$$

Sometimes it is useful to have the phonon Green's function at nonzero temperature. In this case, the thermal average is taken of the phonon occupation numbers,

$$_0\langle|a_{\mathbf{q}} a_{\mathbf{q}}^\dagger|\rangle_0 = N_{\mathbf{q}} + 1 \tag{90}$$

$$_0\langle|a_{\mathbf{q}}^\dagger a_{\mathbf{q}}|\rangle_0 = N_{\mathbf{q}} = \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1} \tag{91}$$

and the Green's function of time is

$$D^{(0)} = (\mathbf{q}, t - t') = -i[(N_{\mathbf{q}} + 1)e^{-i\omega_q|t-t'|} + N_{\mathbf{q}} e^{-i\omega_q|t'-t|}]. \tag{92}$$

8.1 Calculation of Green's function

The Green's function is evaluated by expanding the S matrix $S(\infty, -\infty)$ in Eq. 74 in a series such as Eq. 72:

$$\begin{aligned} G(\mathbf{p}, t - t') \quad &= \quad \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \\ &\quad \frac{_0\langle|T\hat{C}_\lambda(t)\hat{V}(t_1)\hat{V}(t_2)\cdots\hat{V}(t_n)\hat{C}_\lambda^\dagger(t')|\rangle_0}{_0\langle|TS(\infty, -\infty)|\rangle_0} \end{aligned} \tag{93}$$

The terms such as

$$_0\langle|T\hat{C}_\lambda(t)\hat{V}(t_1)\hat{V}(t_2)\hat{V}(t_3)\hat{C}_\lambda^\dagger(t')|\rangle_0 \tag{94}$$

are to be calculated. Suppose that $\hat{V}(t_1)$ is the electron-electron interaction:

$$\hat{V}(t_1) = \frac{1}{2} \sum_{\mathbf{k}'\mathbf{k}\mathbf{q}} \sum_{ss'} \frac{4\pi e^2}{q^2} C_{\mathbf{k}+\mathbf{q},s}^\dagger C_{\mathbf{k}'-\mathbf{q},s'}^\dagger C_{\mathbf{k}',s'} C_{\mathbf{k},s}\, e^{i\,t_1(\xi_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{k}'-\mathbf{q}} - \xi_{\mathbf{k}} - \xi_{\mathbf{k}'})} \tag{95}$$

The *Wick's theorem* states that a time-ordered bracket may be evaluated by expanding it into all possible parirings. Each of these pairings will be a time-ordered Green's function or a number operator n_F or n_B